Comments on "Method of Weighted Residuals Applied to Free Shear Layers"

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USING a method similar to that of Crawford and Holt,[2] in Ref. 1, Stoy obtains an approximation to the solution of the laminar free shear layer flow. We want to draw attention to the following: 1) the mathematical technique Stoy uses is not appropriate for accurate solutions of this class of problem; 2) some of the generally overlooked mathematical aspects of the problem; and 3) an extremely simple method for solving the laminar free shear layer is available.

A clear indication of the difficulty of applying the method of weighted residuals (MWR) in this instance is given by Stoy himself who states that "some experimentation with the numerical solution is necessary . . . every time a new problem is considered." Also, "it was found that the procedure occasionally converged to the wrong answer" unless "the initial guesses were reasonably close to the actual values." After the admitted experimentation to find initial values Stoy inverts matrices to obtain a result within 2% of the published value of one of the less sensitive quantities appearing in the solution. Achievement of higher accuracy is hampered by matrix inversion difficulties which result in "convergence being difficult to observe." In common with a great many users of the MWR technique Stoy does not indicate any considerations of the Schauder basis of the problem and thus eliminates any chance of estimating errors in the numerical solution. A further indication of a certain lack of clarity in describing approximating procedures is displayed when Stoy states that many forms of the approximating function could be used. Actually, the adopted form of $[u(1-u)]^{-1}$ is the only one which gives the proper asymptotic behavior. Alternative choices of the approximating function would probably have resulted in even poorer agreement with published data.

The incompressible momentum equation for the free shear layer is

$$f''' + ff'' = 0 \tag{1a}$$

with the boundary conditions

$$\eta \to -\infty \; f' \to 0; \quad \eta = 0 \quad f = 0; \quad \eta \to +\infty \; f' \to 1 \tag{1b}$$

Here f is the stream function, and primes denote differentiation with respect to the transformed independent variable η. Integration yields

$$f'' = f''(0) \exp\left(-\int_0^\eta f d\bar{\eta}\right) \tag{2}$$

$$\therefore \eta \to -\infty \; f \to -f_o, (f_o > 0) \text{ and } f'' \to \exp(f_o\eta);$$
$$f' \to \exp(f_o\eta) \tag{3a}$$

$$\eta \to +\infty \; f \to \eta \text{ and } f'' \to \exp(-\eta^2/2)$$
$$f' \to 1 - \exp(-\eta^2/2) \tag{3b}$$

The equations show that $f'' \approx f'$ and $f'' \approx 1 - f'$ at the lower and upper boundaries respectively, and thus Stoy's form is the only correct one. At this point it is of interest to note that an exact solution to the free shear layer cannot be obtained numerically and any computing scheme can only approach the mathematical limit. This can be seen if we introduce the Prandtl transposition theorem (Rosenhead[3]) which states that if $U(X,Y)$ is a solution of the boundary layer equations then $u(X,y)$, with $y = Y - g(x)$ is also a solution for an arbitrary $g(x)$. Thus a free shear layer problem is converted to a "blownoff" boundary-layer problem since the transposed boundary conditions are

$$\eta = 0 \quad f' = 0 \quad f = -f_0; \quad \eta \to \infty \quad f' \to 1.0 \tag{4}$$

Now the problem is identical to the one considered by Emmons and Leigh[4] who found that $f''(0) \to 0.0$ as $f_0 \to 0.87574$. The critical value of f_0 for which $f''(0) \equiv 0.0$ cannot be found exactly numerically since all the derivatives at $\eta = 0$ vanish and no numerical integration scheme can be initiated. This problem is discussed by Kassoy[5] who approaches it correctly by means of a parameter expansion technique.

Finally, we shall show that the problem is computationally quite trivial by noting that it belongs to a class of nonlinear integral equations which are easily solved by straightforward iterative techniques (Saaty[6]). The conditions under which the exponential operator in Eq. (2) is a contraction mapping are readily found in any text in modern analysis. Starting with Eq. (2) we generate an iteration sequence:

$$f''(0) = 1 \cdot \Big/ \int_0^\infty e^{\int_0^\eta f d\bar{\eta}} d\eta \tag{5a}$$

$$f'(\eta) = \int_0^\eta f'' d\bar{\eta} \tag{5b}$$

$$f(\eta) = \int_0^\eta f' d\bar{\eta} \tag{5c}$$

$$f(\eta) = f(\eta) - f(\lambda) \tag{5d}$$

The stream function f of Eq. (5d) is used in Eq. (2) and the process is repeated until some convergence criterion is satisfied at $\eta = \lambda$ which corresponds to the dividing streamlines of $f = 0$. The uniqueness and existence of solutions derive from the Banach fixed point theorem (Saaty) and thus Stoy's problem of reaching wrong results is avoided. If $f''(0) = 0.0$ then $f''(\eta) \equiv 0.0$ and therefore only approximate numerical solutions can be obtained. From practical numerical consideration it would be preferable to operate on f'' in such a way that $f''(\lambda)$ would be evaluated in Eq. (5a) but such refinements are not necessary here. The solution was programmed for an IBM 360/65 and as expected from theoretical consideration, it was found that the iteration scheme could be started from an arbitrary initial guess of $f(\eta)$. Trapezoidal integration was considered quite adequate for the present purpose. Integrations were performed over an interval of $\eta = 18$ with $\lambda = 12$. In Table 1 we show the results of a brief study of the effect of integration step size for a solution starting with an initial guess of $f(\eta) \equiv 0$, the most degenerate initial profile.

Table 1 Computed results

Iteration	1	2	3	12	22
			500 steps		
$f''(\lambda)$	0.0571	0.3360	0.2401	0.2828	0.2824
$f'(\lambda)$	0.7120	0.5138	0.6382	0.5868	0.5873
f_o	0	−4.436	−0.5160	−0.8811	−0.8756
			250 steps		
$f''(\lambda)$	0.0571	0.3350	0.2411	0.2828	0.2824
$f'(\lambda)$	0.7080	0.5149	0.6370	0.5869	0.5873
f_o	0	−4.386	−0.5301	−0.8811	−0.8757
			50 steps		
$f''(\lambda)$	0.0571	0.3282	0.2485	0.2828	0.2825
$f'(\lambda)$	0.6800	0.5239	0.6277	0.5873	0.5876
f_o	0	−4.046	−0.6419	−0.8839	−0.8799

Received April 12, 1971.

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No attempt was made to accelerate convergence or to achieve higher accuracy by means of improved integration schemes since only an exhibition of a method rather than accurate solution of a trivial problem is intended. The MWR technique is useful when appropriately applied but the low accuracy achieved by Stoy, with considerable computational difficulties, cannot be considered to be an endorsement of a computing technique in fluid mechanics.

References

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[4] Emmons, H. W. and Leigh, D. C., "Tabulation of the Blasius Function with Blowing and Suction," CP 157, 1954, Aeronautical Research Council, England.

[5] Kassoy, D. R., "On Laminar Boundary Layer Blowoff," *Journal Applied Mathematics*, Vol. 18, No. 1, Jan. 1970, pp. 29–40.

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Reply by Author to A. Wortman and W. J. Franks

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THE comments of A. Wortman and W. J. Franks are well taken and, in general, I agree with them. However, I would like to point out that I have not recommended use of the MWR and, in fact, would not do so for such a trivial problem as the free shear layer. My only purpose in examining the shear layer was to determine some of the specific problems associated with the use of the MWR in situations where no boundary was present. Because of the difficulties in extending the method to complicated flows, such as those with compressibility where some of the dependent variables are double-valued, I do not recommend the use of the MWR over other numerical schemes.

Received May 24, 1971.

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